



New Results on The Stability of Solution of Some Non-autonomous Delay Differential Equations of the Third Order

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Abstract: Sufficient conditions are established for the asymptotic stability of the zero solution of some non-autonomous delay differential equations of the third order. Our result improves on Sadek's [A.I. Sadek, On the stability of solutions of some non-autonomous delay differential equations of the third order, *Asymptotic Analysis* 43(2005) 1-7].

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1. Introduction Consider the third-order nonautonomous delay differential equations

$$\ddot{x} + a(t)\dot{x} + b(t)x + c(t)f(x(t-r)) = 0 \quad (1.1)$$

or its equivalent system form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -a(t)z - b(t)y - c(t)f(x) + c(t) \int_{t-r}^t f'(x(s))y(s)ds, \end{aligned} \quad (1.2)$$

where $a(t), b(t)$ and $c(t)$ are positive and continuously differentiable functions on $[0, \infty)$; r is positive constant; $f(x)$ is continuous function and $f(0) = 0$.

In recent years many books and papers dealt with the delay differential equations and obtained many good results, for example, [1,2,3,6,4,9,5,8,7,10-20], etc. In many references, the authors dealt with the problems by considering Lyapunov functions or functionals and obtained the criteria for the stability.

Recently, Sadek [18] discussed the asymptotic stability of the zero solution of (1.1) and the following result was proved.

Theorem A (Sadek [18]). *Suppose that $a(t), b(t)$ and $c(t)$ are continuously differentiable on $[0, \infty)$ and the following conditions are satisfied:*

- (i) $A \geq a(t) \geq a_0 > 0, B \geq b(t) \geq b_0 > 0, C \geq c(t) \geq c_0 > 0$ for $t \in [0, \infty)$;
- (ii) $f(0) = 0, \frac{f(x)}{x} \geq f_0 > 0 (x \neq 0)$, and $f'(x) \leq f_1 \leq 1$ for all x ;
- (iii) $a_0 b_0 - C > 0$;
- (iv) $\mu a'(t) + b'(t) - \frac{1}{\mu} c'(t) < (a_0 b_0 - C)/2, \mu = (a_0 b_0 + C)/2b_0$;
- (v) $\int_0^\infty |c'(t)| dt < \infty, c'(t) \rightarrow 0$ as $t \rightarrow \infty$.

Then the zero solution of (1.1) is uniformly asymptotically stable, provided that

$$r < \min \left\{ \frac{2c_0 f_0}{f_1 C}, \frac{a_0 b_0 - C}{(1 + a_0) b_0 f_1 C}, \frac{a_0 b_0 - C + 4a_0 C(1 - f_1)}{2f_1 C \{1 + 2\mu + 2a_0^2 + a_0 + (a_0 b_0 - C)C\}} \right\}.$$

Obviously, this is a very interesting result but Theorem A has some hypotheses which are not necessary for the stability of solutions of (1.1).

Our aim in this paper is to further study the stability of the zero solution of (1.1). In the next section, we establish a criterion for the asymptotic stability of the zero solution of (1.1), which extends and improves Theorem A.

Our main result is the following theorem.

Theorem 1.1. *Suppose that $a(t), b(t)$ and $c(t)$ are continuously differentiable on $[0, \infty)$ and the following conditions are satisfied;*

$$(1) \quad 1 \leq c(t) \leq b(t), \quad -L \leq b'(t) \leq c'(t) \leq 0, \quad 0 < a \leq a(t) \leq L, \quad t \in [0, \infty);$$

$$(2) \quad f(0) = 0, \quad \frac{f(x)}{x} \geq \delta_0 > 0 \quad (x \neq 0), \quad \text{and } f'(x) \leq c \text{ for all } x;$$

$$(3) \quad \frac{1}{2}a'(t) \leq \delta_1 < 1 - \alpha c, \quad t \in [0, \infty);$$

$$(4) \quad \int_0^\infty |c'(t)| dt < \infty, \quad c'(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Then the zero solution of (1.1) is uniformly asymptotically stable provided that

$$r < \min \left\{ \frac{2((1 - \alpha c) - \delta_1)}{(2 + \alpha)Lc}, \frac{2(\alpha a - 1)}{\alpha Lc} \right\}.$$

From (first term of) (1), it follows that $b(t)$ and $c(t)$ are non-decreasing functions on $[0, \infty)$ and the limit of each exists as $t \rightarrow \infty$. Since L in (1) is an arbitrary selected bound, we can also assume that

$$1 \leq c(t) \leq b(t) \leq L,$$

$$\lim_{t \rightarrow \infty} c(t) = c_0, \quad \lim_{t \rightarrow \infty} b(t) = b_0 \tag{1.3}$$

$$1 \leq c_0 \leq b_0 \leq L.$$

Remark 1.1. If (1.1) is the constant coefficient delay differential equation $\ddot{x} + a\ddot{x} + \dot{x} + cx(t-r) = 0$, then conditions (1)-(2) reduce to the Routh-Hurwitz conditions $a > 0, c > 0$ and $a > c$. To show this we let $a(t) = a, b(t) = 1$ and

$c(t) = 1$ and $f(x(t-r)) = cx(t-r)$.

2. Preliminaries and Stability Results

We shall in this section give the stability results for (1.1), (hence for system (1.2)). First, we will give the stability criteria for the general non-autonomous delay differential system. We consider

$$\dot{x} = f(t, x_t), \quad x_t = x(t + \theta), \quad -r \leq \theta \leq 0, \quad (2.1)$$

where $f : [0, \infty) \times \mathcal{C}_H \rightarrow \mathbb{R}^n$ is continuous and takes bounded sets into bounded sets and $f(t, 0) = 0$. Here $(\mathcal{C}, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-r, 0] \rightarrow \mathbb{R}^n$ with the supremum norm, \mathcal{C}_H is the open H -ball in \mathcal{C} . Standard existence theory [4] shows that if $\phi \in \mathcal{C}_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ on $[t_0, t_0 + \alpha)$ satisfying (2.1) for $t > t_0$, $x_t(t_0, \phi) = \phi$ and α some positive constant; if there is a closed subset $\mathcal{B} \subset \mathcal{C}_H$ such that the solution remains in \mathcal{B} , then $\alpha = \infty$. Also, (1.1) will denote the norm in \mathbb{R}^n with $|x| = \max_{1 \leq i \leq n} |x_i|$.

We are concerned here with stability in the context of Lyapunov's direct method. Thus, we are concerned with continuous, strictly increasing functions $W_i : [0, \infty) \rightarrow [0, \infty)$ with $W_i(0) = 0$, called wedges, and with Lyapunov functionals, V .

Definition 2.1 [8]. A continuous functional $V : [0, \infty) \times \mathcal{C}_H \rightarrow [0, \infty)$ which is locally Lipschitz in ϕ is called a Lyapunov functional for (2.1) if there is a wedge W with

$$(i) \quad W(|\phi(0)|) \leq V(t, \phi), \quad V(t, 0) = 0, \text{ and}$$

$$(ii) \quad \dot{V}_{(1.2)}(t, x_t) = \limsup_{h \rightarrow 0} \frac{1}{h} \{V(t+h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))\} \leq 0.$$

We have the following fundamental definitions:

Definition 2.2 [8]. (Stability definitions). Since $f(t, 0) = 0$, $x(t) \equiv 0$ is a solution of (2.1) and its is said to be

- (a) stable if for each $\epsilon > 0$ there is a $\delta > 0$ such that $[t \geq 0, \|\phi\| < \delta, t \geq t_0]$ implies that $|x(t, t_0, \phi)| < \epsilon$;
- (b) uniformly stable if for each $\epsilon > 0, t \geq 0$ there is a $\delta > 0$ such that $[\|\phi\| < \delta, t \geq t_0]$ implies that $|x(t, t_0, \phi)| < \epsilon$;
- (c) asymptotically stable if it is stable and if for each $t \geq 0$ there is a $\gamma > 0$ such that $\|\phi\| < \gamma$ implies that $|x(t, t_0, \phi)| \rightarrow 0$ as $t \rightarrow \infty$;
- (d) uniformly asymptotically stable if it is uniformly stable and if there is a $\gamma > 0$ and for each $\mu > 0$ there is a $T > 0$ such that $[t \geq 0, \|\phi\| < \gamma, t \geq t_0 + T]$ implies that $|x(t, t_0, \phi)| < \mu$.

The following is the classical theorem on uniform stability for the solution of (2.1). It goes back to Krasovskii [14].

Theorem 2.1 [7]. *If there is a Lyapunov functional for (2.1) and wedges satisfying:*

$$(i) \quad W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|) \quad \text{and}$$

$$(ii) \quad \dot{V}_{(2.1)}(t, x_t) \leq 0.$$

Then $x = 0$ is uniformly stable.

The basic conjecture for (2.1) on uniform asymptotic stability also goes back to Krasovskii [14] and may be stated as follows.

Theorem 2.2 [7]. *If there is a Lyapunov functional for (2.1) and wedges such that:*

$$(i) \quad W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|) \quad \text{and}$$

$$(ii) \quad \dot{V}_{(2.1)}(t, x_t) \leq -W_3(|x(t)|).$$

Then the zero solution of (2.1) is uniformly asymptotically stable.

3. Proof of Theorem 1.1.

We consider, in place of (1.1) the equivalent system form

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -a(t)z - b(t)y - c(t)f(x) + c(t) \int_{t-r}^t f'(x(s))y(s)ds,\end{aligned}\quad (3.1)$$

and denote $\gamma(t) = \int_0^t |c'(s)|ds$. It may be assumed that $\int_0^\infty |c'(t)|dt \leq N < \infty$. We define the Lyapunov functional $V(t, x_t, y_t, z_t)$ as:

$$V(t, x_t, y_t, z_t) = e^{-\gamma(t)}U(t, x_t, y_t, z_t), \quad (3.2)$$

where

$$\begin{aligned}U(t, x_t, y_t, z_t) &= c(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha\{b(t)y^2 + z^2\} + \alpha c(t)f(x)y \\ &\quad + \frac{1}{2}a(t)y^2 + yz + \lambda \int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds,\end{aligned}\quad (3.3)$$

$\alpha > 0$ is any number chosen such that

$$\frac{1}{c} > \alpha > \frac{1}{a} \quad (3.4)$$

where λ is a positive constant which will be determined later. So that, from (3.3) and (3.1),

$$\begin{aligned}\frac{d}{dt}U(t, x_t, y_t, z_t) &= c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y + \frac{1}{2}a'(t)y^2 \\ &\quad - \{b(t)y^2 - \alpha c(t)f'(x)y^2 - \lambda r y^2\} - \{\alpha a(t) - 1\} z^2 \\ &\quad + c(t)y \int_{t-r}^t f'(x(s))y(s)ds + \alpha c(t)z \int_{t-r}^t f'(x(s))y(s)ds \\ &\quad + \lambda \int_{t-r}^t y^2(\theta)d\theta.\end{aligned}$$

By (2) and using $2uv \leq u^2 + v^2$, we obtain

$$c(t)y \int_{t-r}^t f'(x(s))y(s)ds \leq \frac{1}{2}Lc r y^2 + \frac{1}{2}Lc \int_{t-r}^t y^2(s)ds$$

and

$$\alpha c(t)z \int_{t-r}^t f'(x(s))y(s)ds \leq \frac{1}{2}\alpha Lcrz^2 + \frac{1}{2}\alpha Lc \int_{t-r}^t y^2(s)ds.$$

Therefore

$$\begin{aligned} \frac{d}{dt}U(t, x_t, y_t, z_t) &\leq c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y + \frac{1}{2}a'(t)y^2 \\ &\quad - \left\{ b(t) - \alpha c(t)f'(x) - \frac{1}{2}(Lc + 2\lambda)r \right\} y^2 \\ &\quad - \left\{ \alpha a(t) - 1 - \frac{1}{2}\alpha Lcr \right\} z^2 \\ &\quad + \left\{ \frac{1}{2}Lc(1 + \alpha) - \lambda \right\} \int_{t-r}^t y^2(\theta)d\theta. \end{aligned}$$

If we take $\lambda = \frac{Lc(1 + \alpha)}{2} > 0$, we obtain

$$\begin{aligned} \frac{d}{dt}U(t, x_t, y_t, z_t) &\leq c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y + \frac{1}{2}a'(t)y^2 \\ &\quad - \left\{ b(t) - \alpha c(t)f'(x) - \frac{1}{2}Lc(2 + \alpha)r \right\} y^2 \\ &\quad - \left\{ \alpha a(t) - 1 - \frac{1}{2}\alpha Lcr \right\} z^2. \end{aligned}$$

By (1)-(3), we obtain

$$\begin{aligned}
 \frac{d}{dt}U(t, x_t, y_t, z_t) &\leq c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \\
 &\quad - \left\{ c(t) \left[\frac{b(t)}{c(t)} - \alpha f'(x) \right] - \frac{1}{2}a'(t) - \frac{1}{2}(2 + \alpha)Lcr \right\} y^2 \\
 &\quad - \left\{ \alpha a(t) - 1 - \frac{1}{2}\alpha Lcr \right\} z^2 \\
 &\leq c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \\
 &\quad - \left\{ (1 - \alpha c) - \delta_1 - \frac{1}{2}(2 + \alpha)Lcr \right\} y^2 \\
 &\quad - \left\{ \alpha a - 1 - \frac{1}{2}\alpha Lcr \right\} z^2.
 \end{aligned}$$

If we choose

$$r < \min \left\{ \frac{2[(1 - \alpha c) - \delta_1]}{(2 + \alpha)Lc}, \frac{2(\alpha a - 1)}{\alpha Lc} \right\},$$

we have that there exists $\delta_2 > 0$ such that

$$\frac{d}{dt}U(t, x_t, y_t, z_t) \leq c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y - \delta_2(y^2 + z^2).$$

Next, we show that

$$c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \leq 0$$

for all x, y and $t \geq 0$. From (1), $-L \leq b'(t) \leq c'(t) \leq 0$ for $t \geq 0$, if $c'(t) = 0$, then

$$c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y = \frac{1}{2}\alpha b'(t)y^2 \leq 0$$

since $b'(t) \leq 0$. For those t 's such that $c'(t) < 0$, we have

$$\begin{aligned}
 &c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \\
 &= \frac{1}{2}\alpha c'(t) \left\{ 2\alpha^{-1} \int_0^x f(\xi)d\xi + \frac{b'(t)}{c'(t)}y^2 + 2f(x)y \right\}
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2}\alpha c'(t) \left\{ 2\alpha^{-1} \int_0^x f(\xi)d\xi + y^2 + 2f(x)y \right\} \\ &= \frac{1}{2}\alpha c'(t) \left\{ (y + f(x))^2 + 2\alpha^{-1} \int_0^x \{1 - \alpha f'(\xi)\}f(\xi)d\xi \right\}. \end{aligned}$$

By hypotheses (2) and (3), $1 - \alpha f'(x) \geq 1 - \alpha c > 0$, in view of (3.4); and so using hypothesis (2), we find that

$$2\alpha^{-1} \int_0^x \{1 - \alpha f'(\xi)\}f(\xi)d\xi \geq \alpha^{-1}(1 - \alpha c)\delta_0 x^2.$$

Hence

$$c'(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \leq 0.$$

Therefore

$$\frac{d}{dt}U(t, x_t, y_t, z_t) \leq -\delta_2(y^2 + z^2). \tag{3.5}$$

Since

$$\begin{aligned} &c(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha\{b(t)y^2 + z^2\} + \alpha c(t)f(x)y + \frac{1}{2}a(t)y^2 + yz \\ &= \frac{1}{2}c(t) \left\{ 2 \int_0^x f(\xi)d\xi + \alpha \frac{b(t)}{c(t)}y^2 + 2\alpha f(x)y \right\} + \frac{1}{2} \{a(t)y^2 + 2yz + \alpha z^2\}. \\ &\geq \frac{1}{2}c(t) \left\{ 2 \int_0^x f(\xi)d\xi + \alpha y^2 + 2\alpha f(x)y \right\} + \frac{1}{2} \{a(t)y^2 + 2yz + \alpha z^2\} \\ &= \frac{1}{2}c(t) \left\{ \alpha(y + f(x))^2 + \int_0^x \{1 - \alpha f'(\xi)\} f(\xi)d\xi \right\} \\ &+ \frac{1}{2} \left\{ a(t) \left(y + \frac{1}{a(t)}z \right)^2 + \frac{1}{a(t)}(\alpha a(t) - 1)z^2 \right\} \end{aligned}$$

By (1)-(3), hence there exists some $\delta_3 > 0$ (small enough) such that

$$c(t) \int_0^x f(\xi)d\xi + \frac{1}{2}\alpha\{b(t)y^2 + z^2\} + \alpha c(t)f(x)y + \frac{1}{2}a(t)y^2 + yz \geq \delta_3(x^2 + y^2 + z^2).$$

Thus,

$$U(t, x_t, y_t, z_t) \geq \delta_3(x^2 + y^2 + z^2) \tag{3.6}$$

Therefore we can find a continuous function $W_1(|\phi(0)|)$ with

$$W_1(|\phi(0)|) \geq 0 \quad \text{and} \quad W_1(|\phi(0)|) \leq V(t, \phi).$$

The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $V(t, \phi) \leq W_2(\|\phi\|)$, is easily verified.

From (3.2), we find

$$\frac{d}{dt}V(t, x_t, y_t, z_t) = e^{-\gamma(t)} \left(\frac{d}{dt}U(t, x_t, y_t, z_t) - |c'(t)|U(t, x_t, y_t, z_t) \right).$$

Using the inequalities (3.5) and (3.6), and the fact that $|c'(t)| \geq 0$, we have

$$\frac{d}{dt}U(t, x_t, y_t, z_t) - |c'(t)|U(t, x_t, y_t, z_t) \leq -\delta_2(y^2 + z^2) - \delta_4(x^2 + y^2 + z^2),$$

therefore, if

$$r < \min \left\{ \frac{2((1 - \alpha c) - \delta_1)}{(2 + \alpha)Lc}, \frac{2(\alpha a - 1)}{\alpha Lc} \right\},$$

we have

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq -\delta e^{\gamma(t)}(x^2, x_y^2 + z^2) \leq -W_3(|x(t)|) \quad \text{for some } \delta > 0 \quad (3.7)$$

As a result of (3.7) and the existence of the functions $W_1(|\phi(0)|)$ and $W_2(\|\phi\|)$ we note that the zero of (3.1) is uniformly asymptotically stable.

Remark 3.1. Clearly, our theorem is an improvement and extension of Theorem A. In particular, from our theorem we see that (iv) assumed in Theorem A is not necessary, and (i) (ii) and (iii) can be replaced by (1),(2) and (3) of Theorem 1.1 respectively, for the uniform asymptotic stability of the zero solution of (1.1).

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